Direct integration method for surface waves on depth dependent flows

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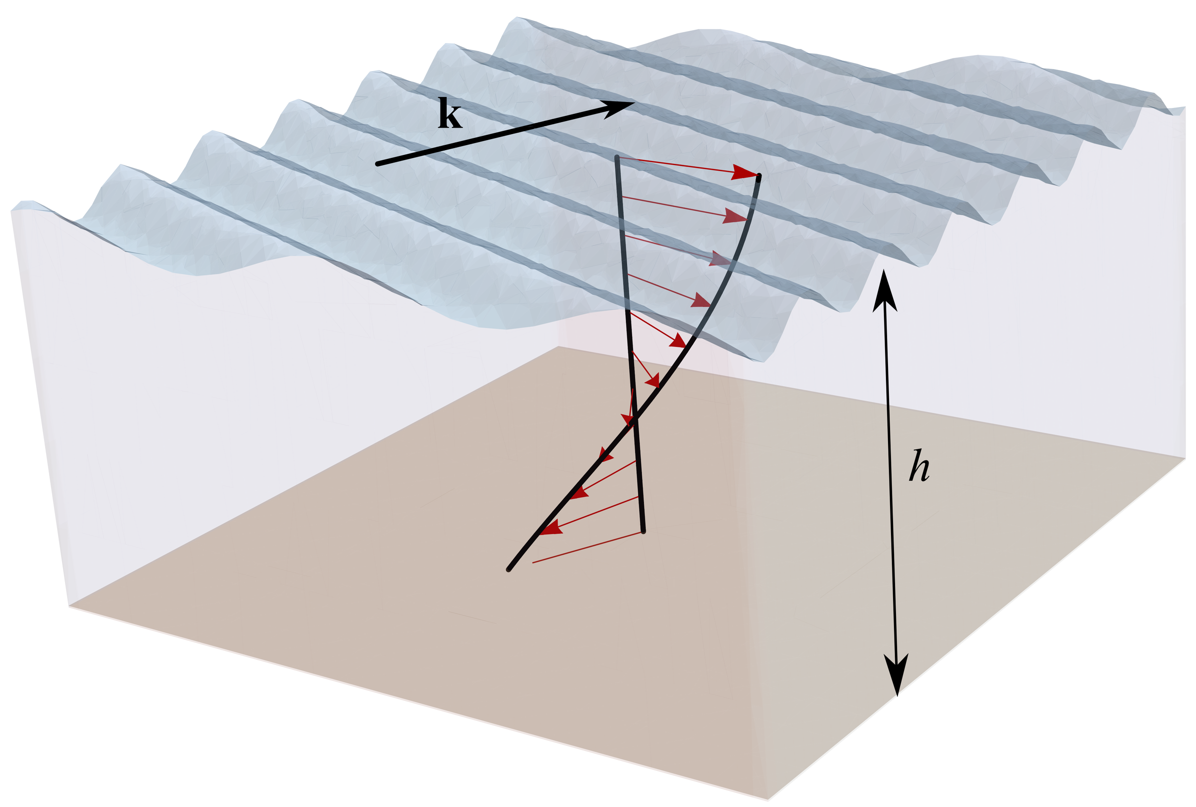
§I-1 Highlights

- A direct numerical method that is
  - computationally cheap
  - with built-in error estimate
  - of arbitrary accuracy
  - easily parallelizeable for many values of wave vector $\mathbf{k}$.
- Applicable to currents of arbitrary depth dependence
- (Arguably) Preferable to existing numerical methods [1,2] for practical purposes
- Computational cost in the same magnitude as that of existing analytical approximations (c.f. e.g. [3,4])
- Full flow field solution with little extra cost

§II Direct integration method



The geometry: a plane wave of wave vector $\mathbf{k}$ travels atop a horizontal background current of arbitrary depth dependence.

We consider a plane wave with wave vector $\mathbf{k} = [k_x, k_y]$ propagating atop a horizontal current $\mathbf{U}(z)$ whose direction and magnitude may vary with depth, running over a flat sea bed of depth h. We seek solutions of a linear wave-current system described by the boundary value problem

$$\bar{w}''(z) - k^2\bar{w}(z) = \frac{\mathbf{k}\cdot\mathbf{U}''(z)}{\mathbf{k}\cdot\mathbf{U}(z) - kc}\bar{w}(z); \qquad (1a)$$

$$(\mathbf{k}\cdot\mathbf{U}(0) - kc)^2\bar{w}'(0) - [\mathbf{k}\cdot\mathbf{U}'(0)(\mathbf{k}\cdot\mathbf{U}(0) - kc) + gk^2 + \frac{\sigma k^4}{\rho}]\bar{w}(0) = 0; \qquad (1b)$$

$$\bar{w}(0) = 1; \qquad (1c)$$

$$\bar{w}(-h) = 0, \qquad (1d)$$

in which the prime is the derivative with respect to z, c is the phase velocity, $\bar{w} = \dfrac{w(\mathbf{k},z)}{w(\mathbf{k},0)}$ (where $w(\mathbf{k},z)$ is amplitude of the vertical velocity due to wave perturbations) is the unity vertical velocity, g is the gravitational acceleration, ρ is the fluid density, and σ is the surface tension coefficient.
Based on Eqs.(1), dispersion relation is obtained

$$D_R(\mathbf{k}, \tilde{c}(\mathbf{k})) \equiv [1 + I_g(\tilde{c})]\tilde{c}^2 + \tilde{c}\mathbf{k}\cdot\mathbf{U}_0' \tanh kh/k^2 - (g/k + \sigma k/\rho)\tanh kh = 0 \qquad (2)$$

in which $\tilde{c} = c - \dfrac{\mathbf{k}\cdot\mathbf{U}(0)}{k}$ and $I_g(\tilde{c}) = \displaystyle\int_{-h}^{0} \mathrm{d}z \frac{\mathbf{k}\cdot\mathbf{U}''\bar{w}(\mathbf{k},z)\sinh k(z+h)}{k(\mathbf{k}\cdot\mathbf{U} - \mathbf{k}\cdot\mathbf{U}(0) - k\tilde{c})\cosh kh}$.

The *direct integration method* uses an iterative approach to solve the linear wave-current system described by the coupled equations (1) and (2) and comes with a built-in error estimate that is introduced in §III.

§III Error estimates

An estimate of the relative error is obtained by a Taylor expansion of (2) about $\tilde{c} = \tilde{c}_{\approx}$ where $\tilde{c}_{\approx}$ denotes an approximation of the exact solution of (2). We obtain

$$R(\tilde{c}_{\approx}) \equiv \left|\frac{\Delta c}{\tilde{c}_e}\right| \approx \left|\frac{D_R(\tilde{c}_{\approx})}{\tilde{c}_{\approx}\frac{\partial D_R}{\partial \tilde{c}}(\tilde{c}_{\approx})}\right|. \qquad (3)$$

§IV Full flow field solution

Once c and $\bar{w}$ are calculated, information of full flow field – that includes amplitude of the dynamic pressure $p(\mathbf{k},z)/\rho$ and amplitudes of the horizontal velocities $u(\mathbf{k},z)$ and $v(\mathbf{k},z)$ – is readily obtained by

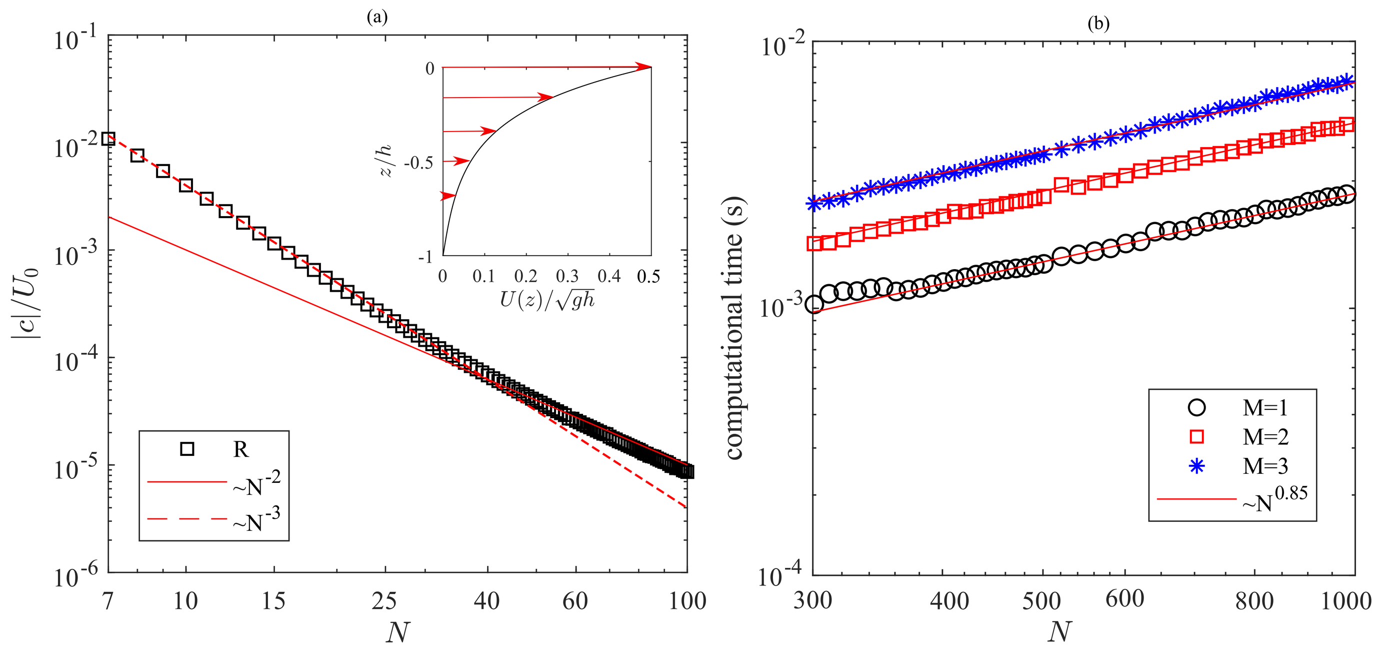
$$\mathrm{i}k^2\bar{p}/\rho = (\mathbf{k}\cdot\mathbf{U} - kc)\bar{w}' - \mathbf{k}\cdot\mathbf{U}'\bar{w}, \qquad (4a)$$

$$k^2(kc - \mathbf{k}\cdot\mathbf{U})\bar{u} = \mathrm{i}k_x[\mathbf{k}\cdot\mathbf{U}'\bar{w} - (\mathbf{k}\cdot\mathbf{U} - kc)\bar{w}'] - \mathrm{i}k^2U_x'\bar{w}, \qquad (4b)$$

$$k^2(kc - \mathbf{k}\cdot\mathbf{U})\bar{v} = \mathrm{i}k_y[\mathbf{k}\cdot\mathbf{U}'\bar{w} - (\mathbf{k}\cdot\mathbf{U} - kc)\bar{w}'] - \mathrm{i}k^2U_y'\bar{w}, \qquad (4c)$$

in which $[\bar{u}, \bar{v}, \bar{p}](\mathbf{k}, z) = [u, v, p/\rho]/w(\mathbf{k}, 0)$.

§ V Convergence and computational time



Demonstration of convergence and calculation time

§VI Comparisons with other approaches

A. Compared to the piecewise-linear approximation [1], the DIM method has several advantages.

- simpler implementation;
- no spurious artefacts of the abrupt change of vorticity at layer interfaces;
- DIM can handle cases where $\mathbf{U}(z)$ varies direction with depth;
- Built-in error estimates.

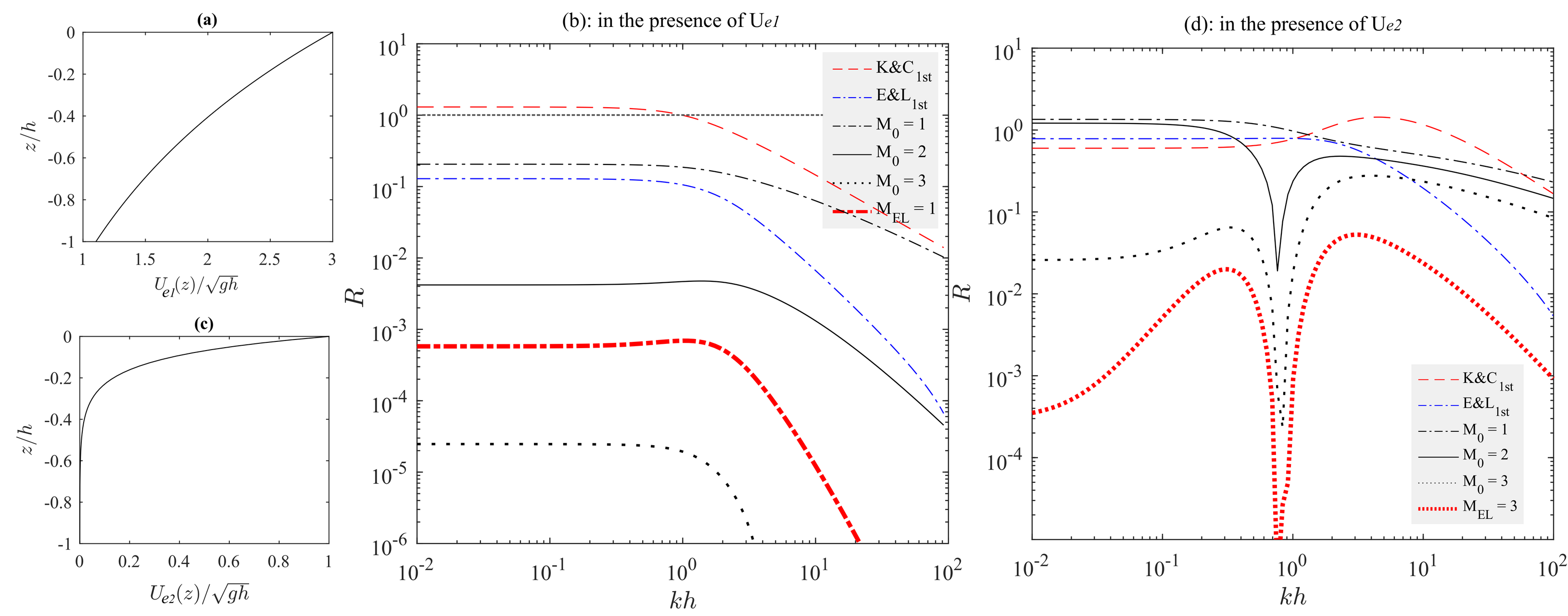
B. Compared to Dong & Kirby's method [2], the DIM method allows

- simpler computations: no nonlinearities introduced;
- parallelizeable computations in an array of wavenumbers for a chosen set of discrete values of z varying from $-h$ to 0;
- calculation of full flow field.

C. Compared to analytical approximations [3,4], the DIM method allows

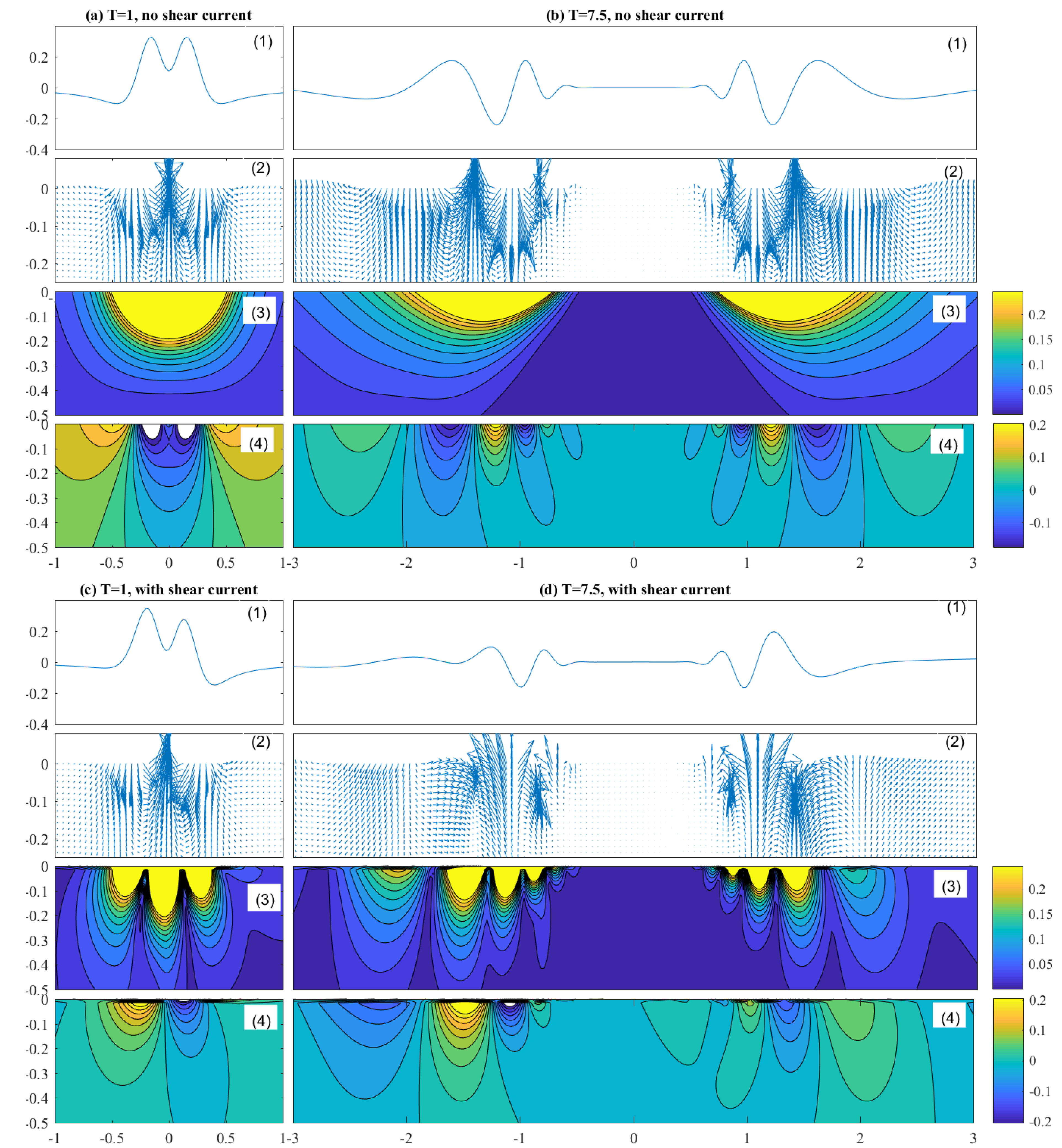
- same accuracy (2-3%) that incurs similar computational cost
- arbitrary accuracy;
- built-in error estimates with little extra cost.

§VII Extreme current profiles



Comparisons among different approximate solutions and the DIM.

§VIII Transient full flow field generated by an initial pressure impulse



Surface elevation, velocity and pressure field in the presence and absence of a shear current at different times.

References

[1] B.K. Smeltzer & S.Å. Ellingsen, Surface waves on arbitrary vertically-sheared currents, *Phys. Fluids* **29**, 047102 (2017).
[2] Z. Dong & J.T. Kirby, Theoretical and numerical study of wave-current interaction in strongly-sheared flows, Coastal Eng. Proc. **33**, waves.2 (2012).
[3] J.T. Kirby & T.M. Chen, Surface waves on vertically sheared flows: approximate dispersion relations, J. Geophys. Res.: Oceans **94**, 1013 (1989).
[4] S.Å. Ellingsen & Y. Li, Approximate Dispersion Relations for Waves on Arbitrary Shear Flows. J. Geophys. Res.- Oceans. **122** (2017).

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