The homogeneous self-compressed compressible sphere: the gravitational overturning and the long period tangential flux

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Compressional and Compositional stratifications

The initial density ρ and bulk modulus κ of spherically symmetric self-gravitating compressible Maxwell Earth are linked by means of the generalized Williamson-Adams equation

$$d_r\rho + \frac{g\,\rho^2}{\kappa} - \lambda = 0 \qquad (1)$$

with λ representing the sum of the no-isentropic temperature (Birch, 1952) and compositional initial density (Kaufmann & Wolf, 2000) gradients. Following Kaufmann & Wolf (2000), I discriminate between compressional, $\lambda = 0$, and compositional, $\lambda \neq 0$, stratifications. The PREM (Dziewonski & Anderson, 1981) is characterized by a compressional lower mantle, while above the 670 km discontinuity it presents compositional stratifications, with both $\lambda > 0$ and $\lambda < 0$. The sign of λ determines the gravitational stability of the models as it results from the following expression for the Brunt-Vaiasala frequency ω

$$\omega^2 = -\frac{g}{\rho}\left(d_r\rho + \frac{g\,\rho^2}{\kappa}\right) = -\frac{g}{\rho}\lambda \qquad (2)$$

As pointed out by Plag & Juttner (1995), stratified models are inherently unstable since

$$d_r\rho = 0 \quad \Longrightarrow \quad \bar\lambda \approx \frac{g\,\rho^2}{\kappa} > 0 \qquad (3)$$

and, so, $\omega^2 < 0$. Actually they present the denumerably set of the unstable RT-modes, which has been interpreted as the gravitational overturning. It has been confirmed by Hanyk *et al.* (1999) which find an approximated analytical expression for the characteristic relaxation times of the RT-modes.

However, I think that stratified compressible models cannot be used to discuss about the gravitational stability since they cannot describe faithfully the type of stratification. It is straightforward to consider that also very fine stratified models based on the PREM overestimate the actual λ of the PREM

$$\bar\lambda \approx \lambda - d_r\rho \qquad (4)$$

where the small differences between the accurate stepwise and continuously variyng parameter profiles have been neglected.

Stratified compressible models cannot describe the compressional and compositional stratifications of the PREM. In particular they are inherently unstable.
In order to investigate the physical processes involved by the different stratifications, I consider a new model of which I have found the analytical solution in the Laplace domain. It is the homoeneous self-compressed compressible (HSC) sphere, which takes into account the continuous variations of the density.

The Homoeneous Self-compressed Compressible (HSC) sphere

At present days the analytical solution of the differential system describing conservation of the momentum and self-gravitation of compressible Maxwell Earth models has been found only for the homogeneous compressible sphere, from now on **HC** sphere (Gilbert & Backus, 1968; Vermeersen *et al.*, 1996). Such a model has been useful to interpret the relaxation spectrum of stratified compressible models but it cannot discriminate between compressional and compositional stratifications since it neglects the continuous variations of ρ. Thus, I define a new compressible model, the homogeneous self-compressed compressible sphere, from now on **HSC** sphere, which can describe both compressional and compositional stratifications and I find its analytical solution in the Laplace domain. It is composed of an inviscid core (Chinnery, 1975) and a viscoelastic mantle with the rheological parameters μ, ν and κ constant. The density ρ profile decreases with the radial distance from the Earth centre r in the mantle and it is constant in the core

$$\rho = \frac{\alpha}{r} \quad (mantle) \qquad \rho_C = \frac{3\,\alpha}{2\,r_C} \quad (core) \qquad (5)$$

Such a ρ profile determines a constant gravity $g = 2\pi G\,\alpha$ in the mantle. The total Earth mass is respected for $\alpha = 2.34\,10^{10}\,kg/m^2$. Further to reproduce the PREM gravity g better than the **HC** sphere, the **HSC** sphere presents also a density contrast $\delta\rho = 3500\,kg/m^3$ at the core-mantle interface comparable with that one of the PREM, $4330\,kg/m^3$. By varying κ the **HSC** sphere describes the different types of stratifications. The κ characterizing the compressional stratification is

$$\kappa_{SC} = 2\pi G\,\alpha^2 \qquad (6)$$

Whatever deviation $\delta\kappa = \kappa - \kappa_{SC}$ implies the following compositional stratification

$$\lambda = \gamma\,\frac{\alpha}{r^2} \qquad with \qquad \gamma = -\frac{\delta\kappa}{\kappa} \qquad (7)$$

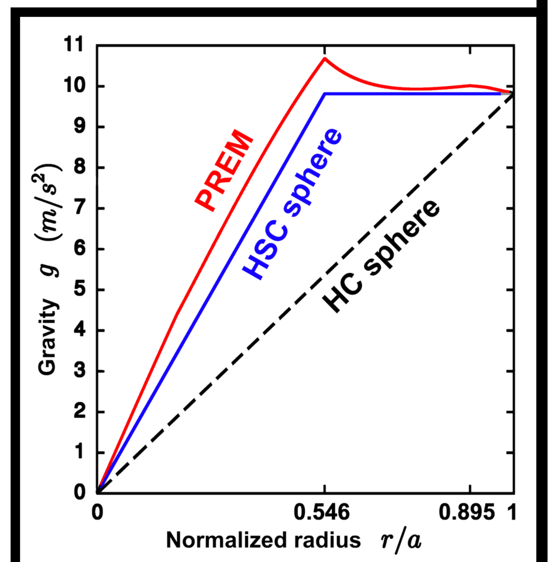

Figure 1. Comparison of the gravity of the homogeneous compressible (**HC**) and homogeneous self-compressed compressible (**HSC**) spheres with the PREM gravity.

which can be stable, $\gamma < 0$, or unstable, $\gamma > 0$. It is possible to show analytically that for compositional stratifications, $\gamma \neq 0$, the secular determinant $\Delta_n(s)$ presents an highly oscillating behaviour

$$\Delta_n(s) \propto \left(\frac{a}{r_C}\right)^{i\left(\frac{\kappa_{SC}\,\gamma\,n(n+1)}{\hat\mu(s)}\right)^{\frac{1}{4}}} \qquad (8)$$

in the limit $s \to 0$. This indicates the presence of a denumerably set of modes that I name compositional C-modes. By setting to zero (8), the pole values s_n^{Cm} of the C-modes result

$$s_n^{Cm} \approx \frac{\tau}{\frac{(m\,\pi)^4\,\mu}{n(n+1)\,\gamma\,\kappa_{SC}\,\log\frac{a}{r_C}} - 1} \qquad (9)$$

Depending on the sign of γ they are stable or unstable. Thus, the C-modes do not describe necessarily the gravitational overturning and, so, I interpret the RT-modes of stratified compressible models as a particular case of the C-modes.

The HSC sphere discriminates between compressional and compositional stratifications. Thanks to its analytical solution, I show the presence of a denumerably set of compositional C-modes, of which the RT-modes are a particular case. Compressional stratifications do not present them.

The relaxation spectrum

Let us consider three **HSC** spheres with $\gamma = 0$ and $\gamma = \pm 0.15$, namely compressional and stable/unstable compositional stratifications. All of them have $\mu = 1.4519\,10^{11}\,Pa$ and $\nu = 10^{21}\,Pa\,s$. The choice of the γ values follows from the indication that the compositional stratification does not exceed the 10-20% of the compressional stratification (Birch, 1952). Further, I consider the 2-layered compressible model **MC**, mantle and core, which differs from $\mathbf{HSC}_{\gamma=0}$ only for the mantle density which is obtained by means of a volume average. Note as all the models presents the C0 and M0 buoyancy modes, the D-modes and the pair of transient compressional modes ($Z0^+$,$Z0^-$). This latter pair of modes is discussed in Cambiotti *et al.* (2009) and, like the D-modes, is associated to the inverse compressional relaxation time

$$\varsigma = \frac{\tau\,\kappa}{\kappa + \frac{4}{3}\mu} \qquad \varsigma = 2.67\,kyr^{-1}\;(\gamma = -0.15) > 2.49\,kyr^{-1}\;(\gamma = 0) > 2.33\,kyr^{-1}\;(\gamma = 0.15)$$

From this point of view the discrepancies in the transient region of the relaxation spectra are easily explained. They are due to the different ς of the models. The M0 and C0 modes of the three **HSC** spheres are quite similar due to the common density profile. Instead the 2-layered **MC** portraies the M0 and C0 modes in a completely different way essentially because of the density contrasts at the core-mantle interface and Earth surface do not agree with those of the **HSC** spheres. Apart these modes, which are well understood by the geophysical community, the most interesting feature consists in the fact that $\mathbf{HSC}_{\gamma=0}$ does not present any buoyancy modes further to the C0 and M0 modes, while the $\mathbf{HSC}_{\gamma=\pm0.15}$ and **MC** models, characterized by compositional stratifications, present the C-modes and RT-modes. It confirms the approximated analytical expression (9) just derived and the analysis of the gravitational stability based on the sign of ω^2. Already at this stage of the analysis, it results that the continuous variations of ρ are essential to describe correctly the type of stratification and it is important at large time scales, in view of the order of magnitude of the characteristic relaxation times of C-modes. The direct dependence form the compositional stratification of the C-modes suggests that the RT-modes of stratifed compressible models do not describe faithfully the gravitational overturnig since they are due to the wrong modellization of the PREM compositional stratification, as shown in (3) and (4).

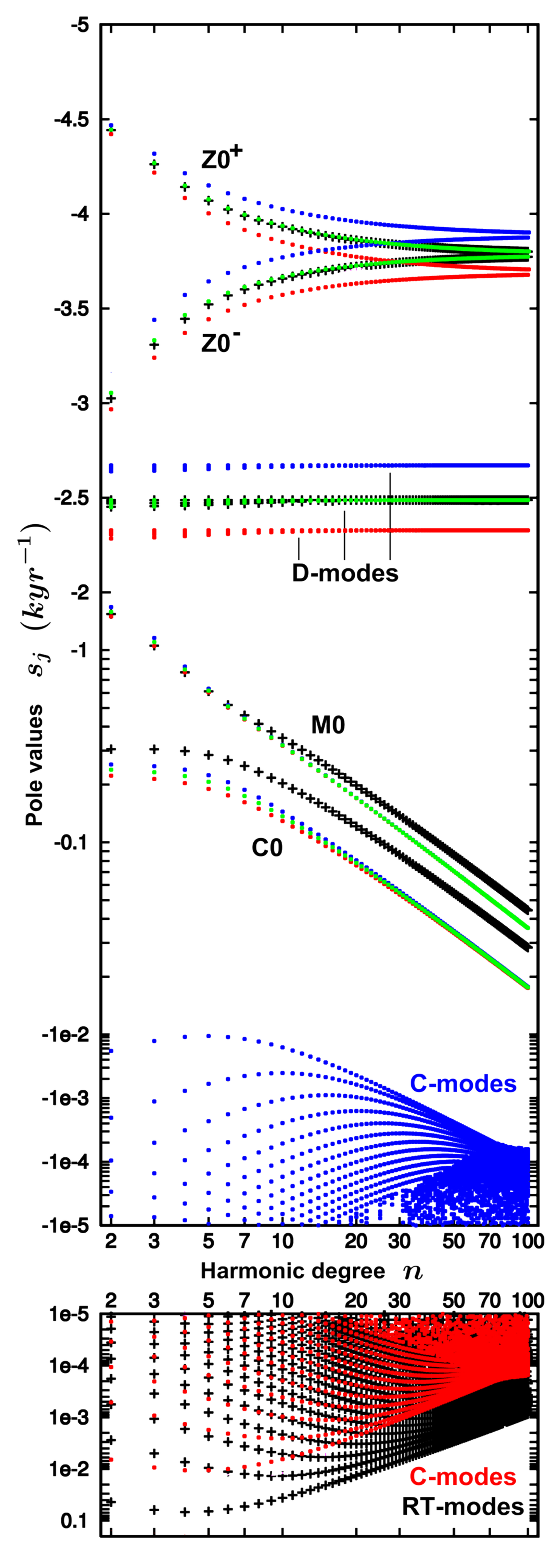

Figure 2. Relaxation spectrum of the 2-layered model **MC (black)** and of the three **HSC** spheres: $\mathbf{HSC}_{\gamma=0}$ **(green)**, $\mathbf{HSC}_{\gamma=-0.15}$ **(blue)** and $\mathbf{HSC}_{\gamma=0.15}$ **(red)**.

The fluid limit

Within the normal mode approach we can express the deformation $\bar k_n(t)$ of the Maxwell Earth models due to an Heaviside point-like load as

$$\bar k_n(t) = \bar k_E - \sum \frac{\bar k_j}{s_j}\left(1 - e^{s_j\,t}\right) \qquad (10)$$

with $\bar k_E$, s_j and $\bar k_j$ being the elastic response, the poles and the residues of the relaxation modes. When the model is stable, the fluid limit $\bar k_n^\infty$ is given by

$$\bar k_n^\infty = \lim_{t\to\infty} \bar k_n(t) = \bar k_E - \sum \frac{\bar k_j}{s_j} = \bar k_n^{ISO} \qquad (11)$$

and agrees with the isostatic response $\bar k_n^{ISO}$ (Chinnery, 1975; Wu & Peltier, 1982). When the model is unstable, $\bar k_n(t)$ diverges in the limit $t \to \infty$, but the agreement between $\bar k_n^{ISO}$ and the summation over $\bar k_E$ and the strengths $\bar k_j/s_j$ holds. It has been pointed out by Hanyk et al. (1999) and Vermeersen & Mitrovica (2000) relatively to stratified compressible models, once the RT-modes are taken into account in the summation, and the same happen to the unstable $\mathbf{HSC}_{\gamma>0}$ sphere with the C-mode. This suggests a slightly different interpretation of the isostatic criterion (Wu & Peltier, 1982). It is not based on a mathematical proof, but on the simple remark that $\bar k_n^{ISO}$ is obtained by elastodynamics by assuming static deformation, while viscolastodynamics deals with quasi-static deformation. Then $\bar k_n^{ISO}$ agrees with the not-time dependent part of the deformation $\bar k_n(t)$

$$\bar k_n^\infty = \bar k_E - \sum \frac{\bar k_j}{s_j} \qquad (12)$$

rather than the fluid limit $\bar k_n^\infty = \lim_{t\to\infty} \bar k_n(t)$. This interpretation reveals a deficiency of the isostatic criterion since the tangential isostatic response l_n^{ISO} results undetermined for models without an elastic outer shell, while the tangential components l_n^∞ of (12) is determined. The same Chinnery (1975) suggested, in the introduction of his paper concerning the elastodynamics, that certain degeneracies in the static solution of the inviscid fluid will be removed, once the viscosity ν would be considered.

The isostatic criterion gives the agreement between the isostatic response and the not time dependent part of the viscoelastic deformation due to an Heaviside forcing.

The long-period tangential flux of material

Let us consider the **HSC** sphere subjected to loading. The l_n^∞ of the compressional $\mathbf{HSC}_{\gamma=0}$ sphere exists and is finite. Instead, for the compositional $\mathbf{HSC}_{\gamma\neq0}$ spheres, the presence of the C-modes arise new problems. Actually the load tangential strengths of the C-modes converge to a finite value

$$\lim_{m\to\infty} \frac{l_n^{Cm}}{s_n^{Cm}} = L_n \neq 0 \qquad (13)$$

instead of decaying rapidly to zero. It means that $l_n^\infty = -\infty$ at each n and, so, the models are tangentially unstable. For the unstable $\mathbf{HSC}_{\gamma>0}$ sphere, this circumstance is not troublesome if compared to the fact that the deformation diverges due to the gravitational overturning. Instead, this is surprisingly for the stable $\mathbf{HSC}_{\gamma<0}$ sphere, since it would seem in constrat with the fact that the model have to achieve a final static state of equilibrium with the load. In order to better clarify this matter, I derive an approximated analytical expression describing the unstable contribution to $l_n(t)$ due to the C-modes of the stable $\mathbf{HSC}_{\gamma<0}$ sphere at large t

$$l_n(t) \approx -\sum \frac{l_n^{Cm}}{s_n^{Cm}}\left(1 - e^{s_n^{Cm}\,t}\right) \approx -\left(\frac{|\gamma|\,t}{\nu}\right)^{\frac{1}{4}} \Gamma\left(\tfrac{3}{4}\right) L_n\,Z_n^{\frac{1}{4}} \qquad (14)$$

by considering the asymptotic patterns

$$\frac{l_n^{Cm}}{s_n^{Cm}} \approx L_n \qquad s_n^{Cm} \approx -\frac{|\gamma|}{\nu}\frac{Z_n}{m^4} \qquad with \qquad Z_n = \frac{n(n+1)\,\kappa_{SC}\,\log\frac{a}{r_C}}{\pi^4}$$

The unstable contribution (14) differs from the exponential growing chacarterizing the gravitational overturning. Moreover, by considering the tangential velocity $d_t\,l_n(t)$, it results that the it goes to zero as $t^{-\frac{3}{4}}$ in the limit $t \to \infty$. Thus, the deformation slows increasing the time, stopping in the fluid limit $t \to \infty$ and, so, the stable $\mathbf{HSC}_{\gamma<0}$ sphere achieves effectively a final static state of equilibrium with the load. I interpret such a deformation as a long-period tangential flux of material. Note as the viscosity ν enters (14), suggesting that it is the viscous friction which slows the flux. Thus, compositional stratification involves the Newtonian linear creep at large time scales more efficiently than compressional stratification.

As it concerns tidal forcing, a similar result hold. Actually also the tidal tangential fluid limit l_n^∞ depends from the type of stratification. This is due to the presence of the C-modes, the tidal tangential strengths of which are greater than the radial and gravitational strengths by 2-7 orders of magnitude. However, in contrast with the load forcing case, the tidal tangential fluid limit l_n^∞ converges to a finite values since the tidal tangential strengths of the C-modes decay to zero rapidly.

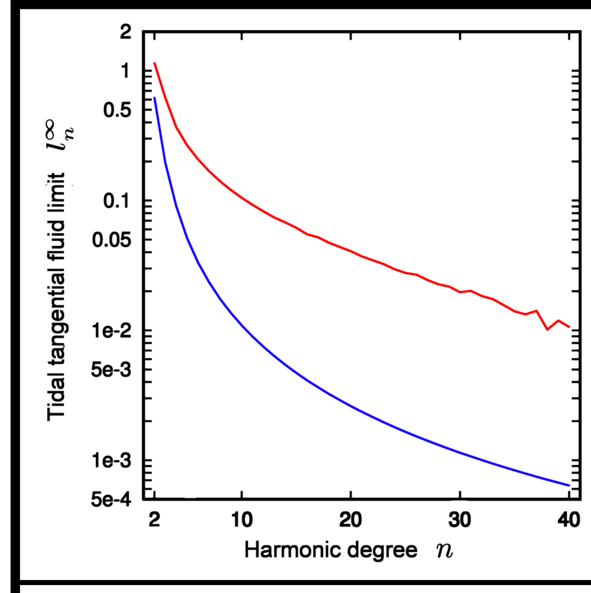

Figure 4. The tidal tangential fluid limit l_n^∞ of the $\mathbf{HSC}_{\gamma=0}$ **(blue)** and $\mathbf{HSC}_{\gamma=-0.15}$ **(red)** spheres

The tangential isostatic response results undetermined for models without an elastic outer shell, while the tangential fluid limit is given by the viscoelastodynamics. Both the load and tidal tangential fluid limits are strongly characterized by the C-modes and, so, they depend from the type of stratification. In particular, the load triggers a long period tangential flux which makes the tangential displacement diverging for compositional stratifications. Compositional stratifications do not present this flux.

Remark The homogeneous self-compressed compressible (**HSC**) sphere, even if it has a continuously varying density profile, can be investigated by means of the normal mode approach. It is in contrast with Han & Whar (1995). On the basis of the large number of modes of very fine stratified compressible models, these authors conclude that continuous variations of model parameter cause dense sets of modes. My results show that this is not necessarily the case since the **HSC** sphere presents at most the denumerably set of C-modes and its density varies continuously within the mantle.

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Figure 3. The load and tidal radial (**diamond**), tangential (**dot**) and gravitational (**cruciform**) strengths $\bar k_j/s_j$ of the compositional $\mathbf{HSC}_{\gamma=\pm0.15}$ spheres at the harmonic degree $n = 2$. The **red** and **blue** colors indicate positive and negative values respectivelly.