Determination of the gradient of curvature of the plumblines of the normal gravity field and the local study of its isocurvature lines

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Introduction

The normal potential of a level ellipsoid is described by the well known formula

$$U(u,\beta)=\frac{GM}{E}\arctan\frac{E}{u}+\frac{1}{2}\omega^2a^2\frac{q}{q_0}\left(\sin^2\beta-\frac{1}{3}\right)+\frac{1}{2}\omega^2\left(u^2+E^2\right)\cos^2\beta$$

where (u, β, λ) are ellipsoidal coordinates. If the normal potential is expressed in a Cartesian system then the curvature of a normal plumb line passing through a point *P* at this specific point is a function of the first and second order partial derivatives of the normal potential *U*. To determine the previous mentioned partial derivatives in a Cartesian system (*X, Y, Z*) we need the partial derivatives of *U* in ellipsoidal coordinates and some suitable matrix transformations. We choose the Cartesian system (*X, Y, Z*) such that the *Z*-axis being the Earth's mean axis of rotation, the *X*-axis is the intersection of the meridian plane of Greenwich and the equator's plane and the *Y*-axis makes the system right-handed. Then we have that

$$\begin{bmatrix}\frac{\partial U}{\partial X}\\ \frac{\partial U}{\partial Y}\\ \frac{\partial U}{\partial Z}\end{bmatrix}=\begin{bmatrix}\frac{u\sqrt{u^2+E^2}}{u^2+E^2\sin^2\beta}\cos\beta\cos\lambda & -\frac{\sqrt{u^2+E^2}}{u^2+E^2\sin^2\beta}\sin\beta\cos\lambda & -\frac{\sin\lambda}{\sqrt{u^2+E^2}\cos\beta}\\ \frac{u\sqrt{u^2+E^2}}{u^2+E^2\sin^2\beta}\cos\beta\sin\lambda & -\frac{\sqrt{u^2+E^2}}{u^2+E^2\sin^2\beta}\sin\beta\sin\lambda & -\frac{\cos\lambda}{\sqrt{u^2+E^2}\cos\beta}\\ \frac{u^2+E^2}{u^2+E^2\sin^2\beta}\sin\beta & \frac{u}{u^2+E^2\sin^2\beta}\cos\beta & 0\end{bmatrix}\begin{bmatrix}\frac{\partial U}{\partial u}\\ \frac{\partial U}{\partial \beta}\\ 0\end{bmatrix}$$

The determination of the second order partial derivatives of *U* in the (*X, Y, Z*) system require the derivation of the first order partial derivatives of *U* in ellipsoidal coordinates and the following transformation

$$\begin{bmatrix}\frac{\partial^2U}{\partial u^2}-\frac{\partial U}{\partial X}\frac{E^2}{\left(u^2+E^2\right)^{\frac{3}{2}}}\cos\beta\cos\lambda-\frac{\partial U}{\partial Y}\frac{E^2}{\left(u^2+E^2\right)^{\frac{3}{2}}}\cos\beta\sin\lambda\\ \frac{\partial^2U}{\partial\beta^2}+\frac{\partial U}{\partial X}\left(u^2+E^2\right)^{\frac{1}{2}}\cos\beta\cos\lambda+\frac{\partial U}{\partial Y}\left(u^2+E^2\right)^{\frac{1}{2}}\cos\beta\sin\lambda+\frac{\partial U}{\partial Z}u\sin\beta\\ 0\\ \frac{\partial^2U}{\partial u\partial\beta}+\frac{\partial U}{\partial X}\frac{u}{\left(u^2+E^2\right)^{\frac{1}{2}}}\sin\beta\cos\lambda+\frac{\partial U}{\partial Y}\frac{u}{\left(u^2+E^2\right)^{\frac{1}{2}}}\sin\beta\sin\lambda-\frac{\partial U}{\partial Z}\cos\beta\\ 0\\ 0\end{bmatrix}=M_1(u,\beta,\lambda)\begin{bmatrix}\frac{\partial^2U}{\partial X^2}\\ \frac{\partial^2U}{\partial Y^2}\\ \frac{\partial^2U}{\partial Z^2}\\ \frac{\partial^2U}{\partial X\partial Y}\\ \frac{\partial^2U}{\partial X\partial Z}\\ \frac{\partial^2U}{\partial Y\partial Z}\end{bmatrix}$$

where $M_1(u, \beta, \lambda)$ is a 6x6 matrix. The second order partial derivatives of *U* are more complicated functions. For the analytical determination of *gradk* also the third order partial derivatives of *U* in Cartesian coordinates are needed and this means far more tedious differentiations. Here we will describe a method to determine approximately the *gradk* without using the third order partial derivatives of the normal potential.

Methodology

Suppose that *P* is a point above the ellipsoid with coordinates (φ_P, λ_P, h_P) and we want to determine a function for the *gradk*. As a first step we introduce the following transformations

$$\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}a\cos\phi_P\cos\lambda_P\\ a\cos\phi_P\sin\lambda_P\\ b\sin\phi_P\end{bmatrix}+\begin{bmatrix}-\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda\\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda\\ 0 & \cos\phi & \sin\phi\end{bmatrix}\begin{bmatrix}x\\y\\h\end{bmatrix}$$

This transformation defines a local Cartesian system (*x*, *y*, *h*) such that the *x* – axis is tangent to the local parallel, the *y* – axis is tangent to the local meridian, and the *h* – axis is the vertical line to the ellipsoid passing through *P*.

The second transformation is

$$\begin{bmatrix}x_1\\y_1\\h_1\end{bmatrix}=\begin{bmatrix}0\\0\\-h_P\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\h\end{bmatrix}$$

which defines a local Cartesian system (x_1, y_1, h_1) which is a parallel transport of the previous one and its center is at the point *P*. The second step is to introduce a set *D* which is in the interior of a unit circle on the meridian plane of *P* which has its center at the point *P* and radius ε. The equation of this circle is $y_1^2+h_1^2=\varepsilon^2$ and ε is equal to one meter. If $\bar{a}_1(t)=(0,y_1(t),h_1(t)),\ t\in[0,\ t_o]$ is the vector equation for the segment of a plumbline passing through a known point of the set *D*, then for the tangent vector and the prime vertical of this segment of the plumbline it holds that

$$\frac{dy_1}{dt}=\frac{\partial U}{\partial y_1}\qquad \frac{dh_1}{dt}=\frac{\partial U}{\partial h_1}$$

and

$$\frac{d^2y_1}{dt^2}=\frac{\partial^2U}{\partial y_1^{\,2}}\frac{\partial U}{\partial y_1}+\frac{\partial^2U}{\partial y_1\partial h_1}\frac{\partial U}{\partial h_1}\qquad \frac{d^2h_1}{dt^2}=\frac{\partial^2U}{\partial y_1\partial h_1}\frac{\partial U}{\partial y_1}+\frac{\partial^2U}{\partial h_1^{\,2}}\frac{\partial U}{\partial h_1}$$

Hence since the curvature of the plumbline *k* is defined as $k(t)=\left|\frac{d\bar{a}}{dt}\times\frac{d^2\bar{a}}{dt^2}\right|\Big/\left|\frac{d\bar{a}}{dt}\right|^3$

it is easily seen that ***it is possible to determine the value of the plumbline curvature at a specific point without knowing the vector equation of the plumbline***.

In the third step we define the following real functions on the set *D*

$$y_1{'}_a(x_1,y_1,h_1)=\left.\frac{\partial U}{\partial y_1}\right|_{(0,0,0)}+\left.\frac{\partial^2U}{\partial y_1^{\,2}}\right|_{(0,0,0)}y_1+\left.\frac{\partial^2U}{\partial y_1\partial h_1}\right|_{(0,0,0)}h_1$$

$$h_1{'}_a(x_1,y_1,h_1)=\left.\frac{\partial U}{\partial h_1}\right|_{(0,0,0)}+\left.\frac{\partial^2U}{\partial y_1\partial h_1}\right|_{(0,0,0)}y_1+\left.\frac{\partial^2U}{\partial h_1^{\,2}}\right|_{(0,0,0)}h_1$$

and we substitute them into the corresponding terms in the equation of the second order partials shown above.

This gives us $y_1{''}_a$ and $h_1{''}_a$, leading in turn into the approximation of the curvature of the plumbline in the unit circle as

$$k_{1a}(y_1,h_1)=\frac{\left|y_{1a}{'}(y_1,h_1)h_{1a}{''}(y_1,h_1)-y_{1a}{''}(y_1,h_1)h_{1a}{'}(y_1,h_1)\right|}{\left(y_{1a}{'}(y_1,h_1)^2+h_{1a}{'}(y_1,h_1)^2\right)^{3/2}}=\ldots=\frac{\left|c_1y_1+c_2h_1+c_3+c_7y_1^{\,2}+c_8y_1h_1+c_9h_1^{\,2}\right|}{c_4(1+2(c_5y_1+c_6h_1+c_{10}y_1^{\,2}+c_{11}y_1h_1+c_{12}h_1^{\,2}))}$$

where c_i (i=1,2,….,12) are real numbers. Hence it is possible to determine the curvature of a plumbline passing through an arbitrary point *Q* in the set *D* at the specific point. While constructing this equation for the curvature of the plumbline in the unit circle ***we made the assumption that the values of all second order partial derivatives of the normal potential in the set D have a constant value which is equal with their value at the point P***. Since this equation is a differentiable function in the set *D* it is possible to determine its partial derivatives and hence to obtain the gradient

$$gradk\Big|_D^{(x_1,y_1,h_1)}(P)=\left.\left(\frac{\partial k}{\partial x_1},\frac{\partial k}{\partial y_1},\frac{\partial k}{\partial h_1}\right)\right|_P=gradk_{1a}\big|_D(0,0,0)=\left(0,c_1-3c_3c_5,c_2-3c_3c_6\right)$$

Finally the coordinates of *gradk*(*P*) in the global Cartesian system (*X*, *Y*, *Z*) are given by the following matrix formula

$$\begin{bmatrix}0\\c_1-3c_3c_5\\c_2-3c_3c_6\end{bmatrix}=\begin{bmatrix}-\sin\lambda_P & \cos\lambda_P & 0\\ -\sin\phi_P\cos\lambda_P & -\sin\phi_P\sin\lambda_P & \cos\phi_P\\ \cos\phi_P\cos\lambda_P & \cos\phi_P\sin\lambda_P & \sin\phi_P\end{bmatrix}\begin{bmatrix}\left.\frac{\partial k}{\partial X}\right|_P\\ \left.\frac{\partial k}{\partial Y}\right|_P\\ \left.\frac{\partial k}{\partial Z}\right|_P\end{bmatrix}$$

The above 3 x 3 matrix is invertible, therefore the coordinates of *gradk*(*P*) can be easily determined in this global frame.

Isocurvature lines

Suppose that $\bar{a}_{is}=\bar{a}_{is}(t)=(X_{is}(t), Y_{is}(t), Z_{is}(t))$ is a curve in the three dimensional space, $\bar{\xi}$ is a conservative vector field, and $k=k(X, Y, Z)$ is the curvature function for $\bar{\xi}$ i.e. it describes the curvature for its integral curves for a specific point (X_0, Y_0, Z_0). Let k_{is} be the curvature function for $\bar{a}_{is}$, $P=(X_P, Y_P, Z_P)$ is a point lying on $\bar{a}_{is}$ and $\bar{a}=\bar{a}(t)$ *is* the equation of the plumbline of the field passing through *P*. We say that the curve $\bar{a}_{is}$ is an ***isocurvature line*** of $\bar{\xi}$ with ***ā-field strength*** $R_a(X_P,Y_P,Z_P):=k(X_P,Y_P,Z_P)$ if it holds that $(k\circ\bar{a}_{is})(t)=k(X_P,Y_P,Z_P)$.

We can now determine the isocurvature lines of the Earth's normal gravity field passing through a point *P* above the ellipsoid. The curvature of the normal plumbline passing through the point *P* is equal to $k(X_P, Y_P, Z_P)$. Due to the symmetry of the field all the plumb lines which pass from points with coordinates (φ_P, λ, h_P) will have the same value of curvature. The locus of these points is a circle which is an isocurvature line that has the following vector equation

$$\bar{a}_{is}^{I}(\lambda)=\left((N(\phi_P)+h_P)\cos\phi_P\cos\lambda,(N(\phi_P)+h_P)\cos\phi_P\sin\lambda,\left(\frac{b^2}{a^2}N(\phi_P)+h_P\right)\sin\phi_P\right)$$

Since the plumblines of the normal gravity field are plane curves (lying on a meridian plane) then the second isocurvature line is a plane curve and lies on the meridian plane of the point *P*. If we introduce the local Cartesian coordinate system (x_1, y_1, h_1) then the equation of this isocurvature line in the set *D* will be the solution of the equation $k(y_1,h_1)=k(P)$, which needless to say is a very complicated algebraic equation and it cannot be solved analytically. But from the initial value theorem it is possible to determine the direction of the isocurvature line in the set *D*. That is, if the sought solution is

$$a_{is}^{II}=a_{is}^{II}(y_1)\equiv h_1(y_1)$$

then the direction of the isocurvature line at *P* is given by.

$$\left.\frac{da_{is}^{II}}{dy_1}\right|_P=-\left.\frac{\partial k}{\partial y_1}\right|_P\Big/\left.\frac{\partial k}{\partial h_1}\right|_P$$

and the next point Q_1 of the isocurvature line will have coordinates

$$Q_1=\left(0,y_{1_{Q_1}},h_{1_{Q_1}}\right)=\left(0,\left(-\left.\frac{\partial k}{\partial y_1}\right|_P\Big/\left.\frac{\partial k}{\partial h_1}\right|_P\right)\delta h_1,\delta h_1\right)$$

By repeating this procedure it is always possible to determine the direction of the isocurvature line at the next points. However, because, by doing so we do not locate the points of the isocurvature line exactly, after the n-th iteration we have to apply a small correction to the coordinates of the point Q_n, by first determining the value of curvature at the point Q_n, $k(Q_n)$ and subtracting it from $k(P)$, and in a second step, since the difference $k(P) - k(Q_n)$ is very small, estimating the new coordinates of point Q_n as

$$Q^{new}{}_n=\left(0,y_{1_{Q_n}},h_{1_{Q_n}}+[k(P)-k(Q_n)]\Big/\left.\frac{\partial k}{\partial h_1}\right|_{Q_n}\right)$$

The second isocurvature line is vertical to the the first and it has two ending points on the surface of the ellipsoid. The first conclusion is obvious since the first isocurvature line lies on a plane parallel to the equatorial plane and the second lies on the meridian plane of the point *P*. For the second conclusion, let point *P* be outside of the ellipsoid and *k*(*P*) be the value of R_a. ***As the isocurvature line approaches the equatorial plane it intersects plumblines whose curvature near the ellipsoid becomes smaller and tends to zero***. The plumbline which is on the meridian plane $\lambda = \lambda_P$ and has the point $(\varphi, \lambda) = (0, \lambda_P)$ as a "starting" point is a straight line, and hence it has curvature equal to zero. Furthermore, ***since the isocurvature line intersects plumblines with non-zero curvature*** (and the field is stronger as we move towards the surface of the ellipsoid), ***this means that it approaches the ellipsoid and one of the last points will be on the surface of the ellipsoid***. The same situation holds as we approach the north pole, and therefore ***the isocurvature line has the second of its ending points on the ellipsoid near the north pole***.

Due to the symmetry of the level ellipsoid the situation is depicted in Figure 1.

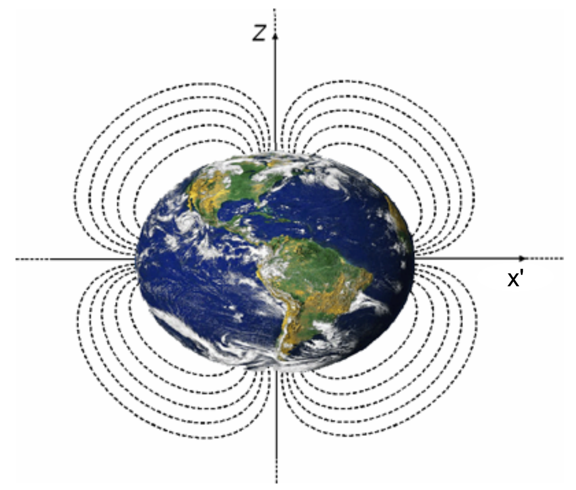

Figure 1

Breaking of an isocurvature line

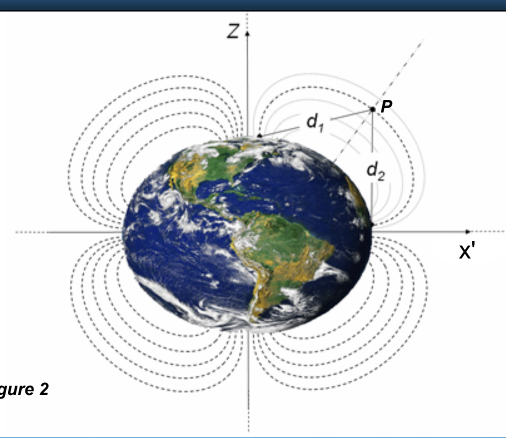

Figure 2

The line segments d_1 and d_2 *connecting the two end points with the point P* are called ***diameters*** of the isocurvature line. As the geometric height of *P* tends to infinity (φ_P and λ_P remain constant), the two ending points approach asymptotically the equator and the north (or south) pole respectively. If the point *P* "reaches infinity" then the isocurvature line "breaks" in two pieces, one along the axis *x'* and the other along the axis *Z* (as shown in Fig. 2). The first piece is a straight line vertical to the ellipsoid an lies on the equatorial plane and the second is a vertical line to the ellipsoid at the north (or south) pole. Considering that these straight lines are plumblines of the normal gravity field we can further state the following:

- **In case of R_a = *0* the isocurvature lines are also plumblines of the normal gravity field**
- **Finite diameter length of the isocurvature line means non zero curvature for the plumblines**
- **As the curvature of the plumblines tends to zero then d_1 and d_2 tend to infinity**

Since the value of the diameter d_1 depends on the value of d_2, let $d = 0.5(d_1+d_2)$. From the above conclusions we can further state that

On the meridian plane there is a connection between the curvature of the plumblines k (or R_a) and the value of d and it has the form $k = k(1/d)$

Isocurvature surfaces

Suppose that

$$\bar{s}_{is}:D_1\to\Re^3:(u,v)\to\bar{s}_{is}(u,v)=\left(X_{is}(u,v),Y_{is}(u,v),Z_{is}(u,v)\right)$$

is a vector equation for a coordinate patch of a surface *S* in the three dimensional space.

Let $\bar{\xi}$ be a conservative vector field and $k=k(X, Y, Z)$ is the curvature function and $P = (X_P, Y_P, Z_P)$ is a point on *S*. We say that the surface *S* ***is an isocurvature surface of*** $\bar{\xi}$ with ***ā-field strength*** $R_a(X_P,Y_P,Z_P):=k(X_P,Y_P,Z_P)$ if the following relation holds

$$\left(k\circ\bar{s}_{is}\right)(u,v)=k\left(X_P,Y_P,Z_P\right)$$

For the normal gravity field holds that the isocurvature surface passing through the point *P* is a surface of revolution. Therefore, since the field is symmetric, let *P* be on the *XZ* – plane. The isocurvature line which lies on the meridian plane of *P* is possible to have a local parametric representation of the form

$$\bar{a}_{is}^{II}(\phi)=\left(f_1(\phi),0,f_2(\phi)\right)$$

In other words, ***the isocurvature surface passing through a point P - due to the symmetry of the field – is generated from the rotation of the isocurvature curve along the Z – axis***, hence a parametric representation for a coordinate patch of the isocurvature surface passing through the point *P* in (φ, λ) coordinates has the form

$$\bar{s}_{is}^{P}(\phi,\lambda)=\left(f_1(\phi)\cos\lambda,f_1(\phi)\sin\lambda,f_2(\phi)\right)$$

In summary,

- Since we proved that there is one isocurvature surface passing through any point *P* above the ellipsoid, it means that ***there are infinite isocurvature lines passing through the point P***, all lying on the surface S.
- In turn, since these isocurvature lines can be thought of as covering (at least locally) the surface S, ***for any direction there is always one isocurvature line***. Mathematically, this means that ***the family of these isocurvature lines passing through P can be described from a 2nd order system of ordinary differential equations***.
- In addition, since ***the isocurvature line on the meridian plane is the generating curve for the isocurvature surface*** we define the isocurvature line on the meridian plane as ***the mother isocurvature line***.

Conclusions

We have outlined a method for determining the gradient *gradk* of the curvature of the plumblines of the Earth's normal gravity field at a point *P* without using the (ordinarily required) third order partial derivatives of the normal potential *U*. With this method extra derivations and inversion of complicated matrices is avoided. The hypothesis about the second partial derivatives of *U* was that they do not change in the interior of a unit circle (i.e. of radius ε = 1 *m*) whose center was the point of interest *P*. This methodology is useful for the determination of the isocurvature lines passing through any point *P*. Along these lines the *ā-field strength* R_a of the normal gravity field is constant and its value $R_a(P)$ is equal with *k*(*P*). We proved that there are at least two isocurvature lines passing through a point *P,* one on the meridian and the other on the parallel plane of *P*, and hence they are perpendicular to one another. Both isocurvature lines represent a family of isocurvature lines as *P* changes its position above the ellipsoid, with only restriction that *P* is on a meridian plane. The first family are circles which lie on planes which are parallel to the equatorial plane and the second family are curves on the meridian plane which have two ending points on the surface of the ellipsoid.

The second family has more interest because, as the geometric height of the point *P* tends to infinity then the isocurvature line tends asymptotically to the *x'*-axis on the equatorial plane and the *Z*-axis, and subsequently "breaks" into two pieces. These two pieces have the special property that they are plumblines of the normal gravity field and they have constant value of curvature equal to zero.

Finally we proved that they are infinite isocurvature lines passing through any point *P* and they lie on a special surface which is called an isocurvature surface of the normal gravity field. The isocurvature lines and isocurvature surfaces are new geometric objects whose geometric properties may reveal new properties for the normal gravity field and thus contribute to a better understanding of the Earth's normal gravity field.