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Geometric aspects and meteorological applications of Nambu mechanics: The motion of three point vortices in the plane

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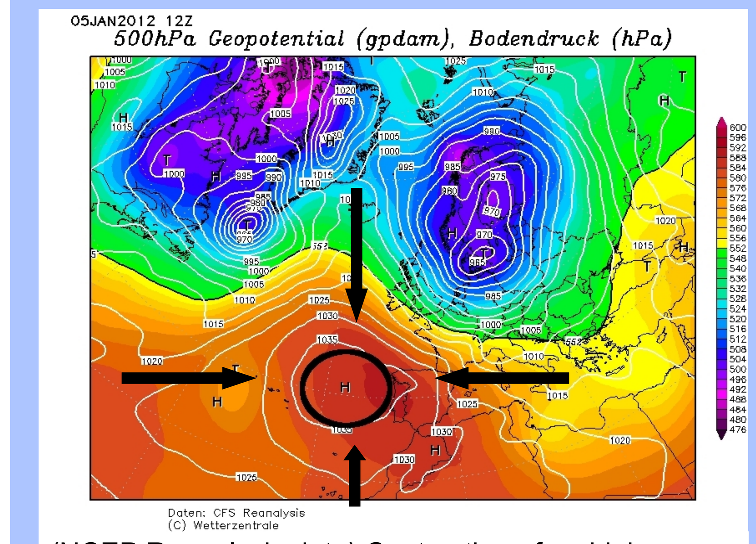
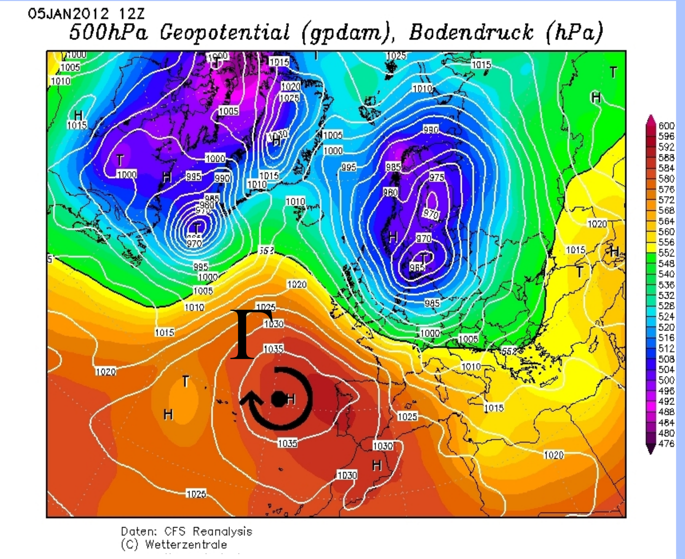
Introduction

On synoptic scale point vortices can be seen as contracted high and low pressure areas determined by their circulation Γ.

$\Gamma > 0$: cyclonic rotation
$\Gamma < 0$: anticyclonic rotation



(NCEP Reanalysis data) Contraction of an high pressure area



The point vortex is characterised by the circulation Γ

Compared to the classical representation of three point vortices, the application of Nambu mechanics leads to a geometrical representation of dynamics in a three dimensional phase state. In general, point vortex dynamics can be seen as discrete form of the 2D barotropic and incompressible vorticity dynamic.

Summary and Outlook

Theoretically, describing the motion of three point vortices in terms of Nambu mechanics leads to a reduction of the dimension of the phase state. The trajectory is given by the intersection line of two sufaces represented by two conserved quantities.
A meteorological application of three point vortices is the so-called Omega-Blocking. The comparison of analytical results with reanalysed ERA-INTERIM-data show that the stationarity of blocking events can be described by three point vortices.
It would be interesting to apply point vortices to different atmospheric flows.

Classical representation
Helmholtz (1858)/ Kirchhoff (1876)

VS

- 6 dimensional phase space:

$$\vec{x} = (x_1, y_1, x_2, y_2, x_3, y_3)$$

- 6 differential equations:

$$\frac{dy_i}{dt} = +\frac{1}{2\pi} \frac{\sum_{i,i=1}^{3} \Gamma_i (x_i - x_j)}{r_{ij}^2}$$

$$\frac{dx_i}{dt} = -\frac{1}{2\pi} \frac{\sum_{i,j=1}^{3} \Gamma_i (y_i - y_j)}{r_{ij}^2}$$

with relative distances:

$$r_{ii}^2 = (x_i - x_i)^2 + (y_i - y_i)^2, \quad i, j = 1,2,3$$

Nambu representation
Nambu (1973)/ Névir (1998)

- 3 dimensional phase space:

$$\vec{r} = (r_{12}, r_{23}, r_{31})$$

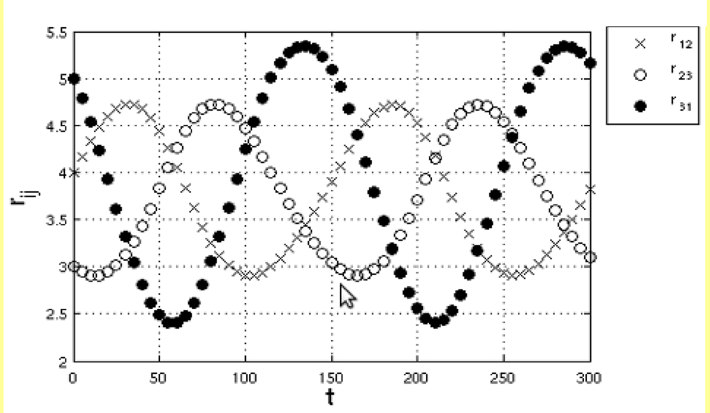
- 3 differential equations

$$\rho \frac{d\vec{r}}{dt} = \nabla M \times \nabla H$$

$$\rho = \frac{r_{12} r_{23} r_{31}}{4A}, \quad T = \frac{t}{2\Gamma_1 \Gamma_2 \Gamma_3}$$

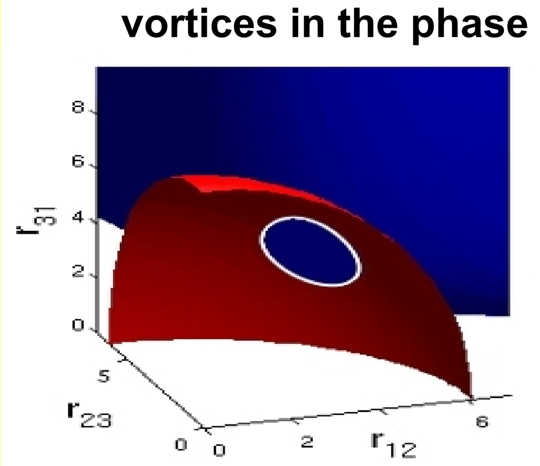
Trajectory as intersection line of M- and H-surface

Classical representatin of periodic motion



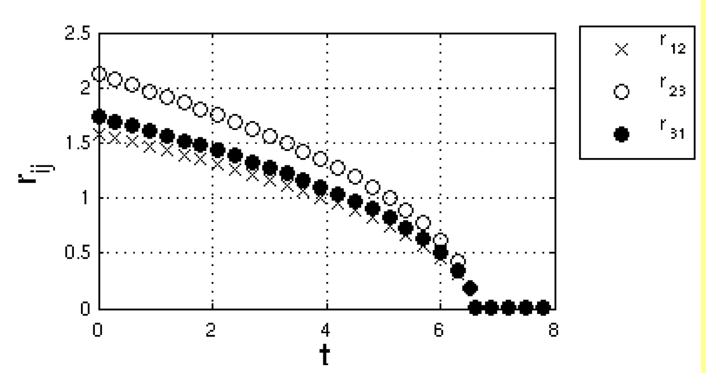
Periodic motion of three point vortices

Periodic motion of three point vortices in the phase space



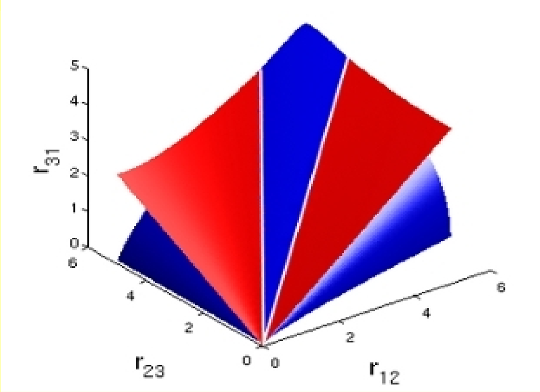
The M-surface represents an ellipsoid (red), therefore the trajectory is given by a closed line

Classical representation of collapse motion



Collapse motion of three point vortices

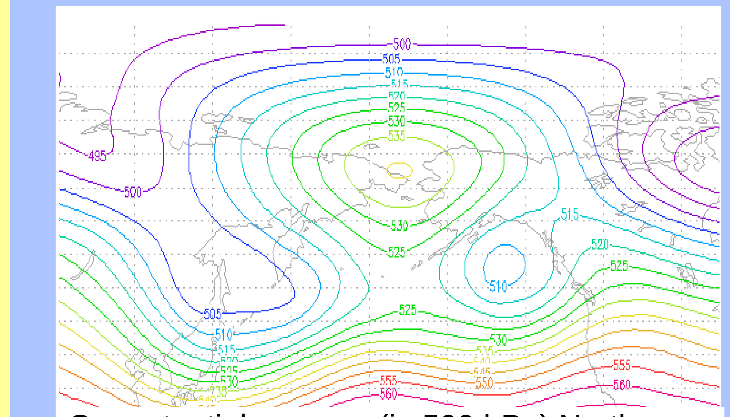
Collapse motion of three point vortices in the phase space



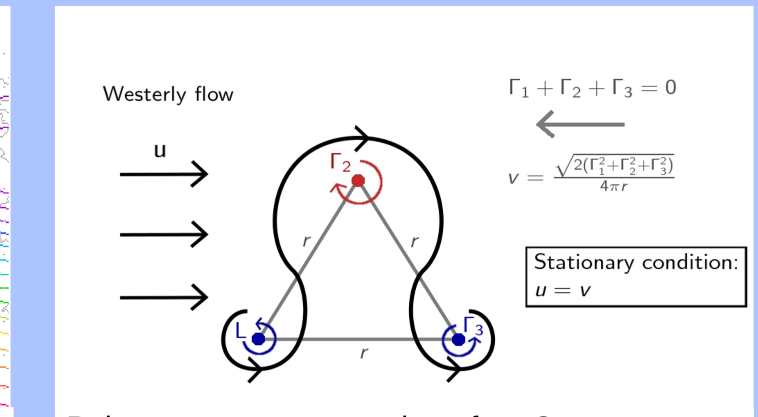
The M-surface is part of a cone (red), therefore the trajectory is given by two closed lines

Meteorological Application: Omega-Blocking

Synoptic motions can be described by geostrophic wind fields on isentropic surfaces. Therefore, they can be considered as 2D incompressible flows leading to an application of point vortex theory. As example we analyse a persistent large scale weather situation, the so-called Omega-Blocking.



Geopotential average (in 500 hPa) North Pacif c, 1.-12.3.2011



Point vortex representation of an Omega-Blocking

The geometric evaluated **total circulation** Γ_{ges} is given by:

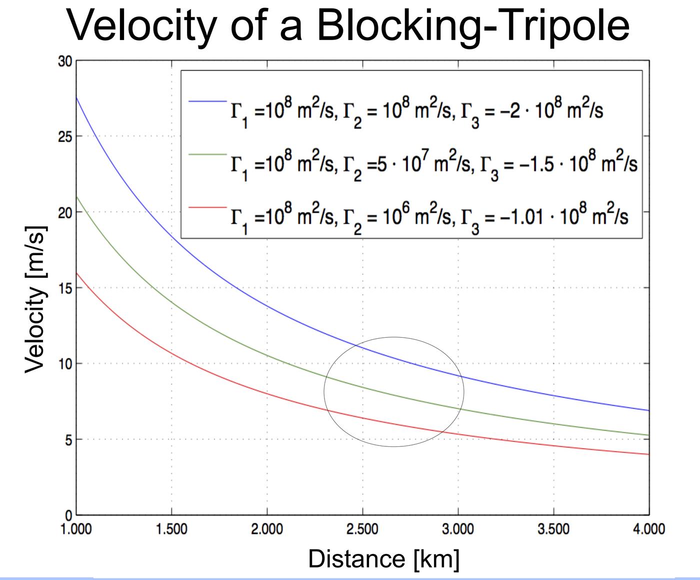
$$\Gamma_{ges} = \Gamma_1 + \Gamma_2 + \Gamma_3 = 0.03\, 10^8\, m^2/s,\; r = 4000\text{km}$$

The **translation velocity** ***v*** (east to west) can be calculated by point vortex theory::

$$v = (\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2)^{1/2} / 4\pi r^2 = 12\, m/s$$

The **basic flow** ***u*** (west to east) was calculated by averaging of 48 time steps of reanalysed data in the mean geographic latitude $(54 \pm 15)^\circ$N: $u = 15\text{m}/s$

The figure on the right hand side illustrates analytical solutions of the theoretical determinded velocity of three point vortices forming an equilateral triange. Meteorologically, Γ_1 and Γ_2 represent two low pressure areas and Γ_3 one high pressure area. As in the above application, at about 2000-3500 kilometers, the wind velocity is 5-10m/s which coincides with well known wind velocities in synoptic scale.



Acknowledgements
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Refereces
[1] Helmholtz H. 1858: *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen* J.Reine Angew. Math. **55**, 25-55
[2] Kirchhoff G. R. 1876: *Vorlesungen über Mathematische Physik Vol.* ***1*** *Leipzig, Teubner*
[4] Nambu Y. 1973: Generalized Hamiltonian Dynamics *Phy. Rev.* **D 7,** 2405-2412
[4] Névir P. 1998: Die Nambu Felddarstellungen Hydro-Thermodynamik und ihre Bedeutung für die dynamische Meteorologie, Hab. Diss., FU Berlin

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Basic Quantities:

Circulation

$$\Gamma = \int_{\delta A} \vec{v} \cdot d\vec{r} = \int_A rot\, \vec{v}\, d\vec{A}$$

Total Circulation

$$\Gamma_{ges} = \sum_{i=1}^{N} \Gamma_i$$

Zonal Momentum

$$P_y(x_i, y_i) = -\sum_{i=1}^{N} \Gamma_i x_i$$

Meridional Momentum

$$P_x(x_i, y_i) = \sum_{i=1}^{N} \Gamma_i y_i$$

Angular Momentum

$$L_z(x_i, y_i) = -\frac{1}{2} \sum_{i=1}^{N} \Gamma_i (x_i^2 + y_i^2)$$

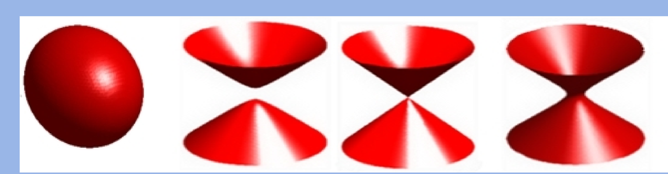
Center of Circulation

$$\vec{x}_C = \frac{\sum_{i=1\neq}^{N} \Gamma_i \vec{x}_i}{\sum_{i=1}^{N} \Gamma_i}, \quad \vec{x}_i = (x_i, y_i)^T$$

Relative Angular Momentum and Total Energy

$$M = -Z L_z - \frac{1}{2}(P_x^2 + P_y^2) = \frac{1}{4} \sum_{i \neq j, i, j=1}^{N} \Gamma_i \Gamma_j r_{ij}$$

$$M \propto \frac{r_{12}^2}{\Gamma_1} + \frac{r_{23}^2}{\Gamma_1} + \frac{r_{13}^2}{\Gamma_2}$$



The M-surface represents an ellipsoi, a two sheeted hyperboloid, a cone or an one sheeted hyperboloid

$$H = -\frac{1}{4\pi} \sum_{i \neq j, i, j=1}^{N} \Gamma_i \Gamma_j \ln(r_{ij})$$

$$H \propto \ln \frac{(r_{12})}{\Gamma_1} + \ln \frac{(r_{23})}{\Gamma_1} + \ln \frac{(r_{13})}{\Gamma_2}$$